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BRST₁ QUANTIZATION OF THE GREEN–SCHWARZ SUPERSTRING

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We apply a one-step Noether extended BRST procedure (BRST₁ procedure) to quantize the Green–Schwarz heterotic string. The gauge fixed action is presented in a wide class of gauges including the unitary light-cone gauge as well as Lorentz-covariant gauges containing an infinite number of ghosts. We show that the BRST₁ transformation rules and the proof of the gauge independence of the theory are simple for a particular choice of variables. We present the explicit (on-shell closed) BRST₁ symmetry of the theory in the recently discovered covariant gauge where the action is quadratic in fields. We find that the BRST₁ transformations of the right- and left-moving modes are separately off-shell nilpotent.

We furthermore point out that in order to show that the gauge fixed action is BRST₁ invariant it is necessary to regularize certain infinite summations over the fields of the theory. This regularization leads to non-trivial integrability conditions which in the case of the BRST₁ symmetry are satisfied. We give examples of other systems with infinite variables where this is not the case.

1. Introduction

Recently, important progress has been made in solving the longstanding problem of covariantly quantizing the Green–Schwarz superstring [1–3]. The idea of a covariant gauge fixing leading to a quadratic gauge fixed action was proposed for the first time in ref. [4] in the context of the superparticle. The discussion of a corresponding gauge fixing for the superstring was presented by Siegel in the Texas A&M meeting [5]. The reason that the covariant quantization of the Green–Schwarz superstring has proved to be difficult is twofold. First of all it turns out that the theory is infinite reducible which requires the introduction of an infinite number of ghosts in covariant gauges. Secondly the theory possesses a fermionic so-called κ -symmetry whose gauge algebra only closes when field equations are used. Both complications were dealt with in refs. [1–3] by using a general method which was

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developed by Batalin and Vilkovisky (BV) [6]. Quite surprisingly, it turns out that although the formulation of the original Green–Schwarz string is rather complicated, the resulting gauge fixed action reduces to a free conformal field theory for a particular Lorentz-covariant gauge and a particular choice of variables.

In this paper we will explain that the simplicity of the above result can be understood within the framework of a one-step Noether extension of the standard BRST procedure. The standard BRST formalism [7] was originally developed to quantize irreducible (in the sense of BV) gauge theories with an off-shell closed algebra, like e.g. Yang–Mills theories. Later it was adapted to include the quantization of finite reducible gauge theories with an off-shell closed algebra as well. Examples of such finite reducible theories are the two-index antisymmetric tensor [8] and general p -forms [9].

It was realized soon that the standard BRST procedure cannot be applied to quantize theories that possess an *on-shell* closed gauge algebra, i.e. theories whose gauge algebra only closes when equations of motion are used. Historically, the first example of such a theory was supergravity without auxiliary fields. The necessary extended BRST procedure to quantize supergravity was developed in 1977 by one of the authors [10]. The procedure presented in ref. [10] is essentially a Noether-type extension of the standard BRST procedure [7]. In supergravity and in many other theories the Noether extension of the standard BRST involves only one step. Henceforth we will denote this one-step Noether extended BRST procedure as the BRST_1 procedure. An essential difference between the standard BRST and the BRST_1 procedure is that the standard BRST transformations close off-shell whereas the BRST_1 transformation rules close on-shell. Later the BRST_1 procedure was generalized by de Wit and van Holten [11] to the more general situation of irreducible theories with an on-shell closed algebra.

In 1983 Batalin and Vilkovisky constructed a unified approach to the quantization of theories with an on-shell closed gauge algebra which at the same time could be irreducible [12], like supergravity, or finite reducible [6]. An example of a finite reducible gauge theory with an on-shell closed algebra is the non-abelian antisymmetric tensor model of Freedman and Townsend [13]. This model has been quantized by Thierry-Mieg [9] using the BRST_1 procedure. Only recently the quantization of the same system using the general formalism of Batalin and Vilkovisky was presented in several papers [14].

It was realized in ref. [15] that the Green–Schwarz superstring is an example of a theory that does not only have an on-shell closed gauge algebra but is also *infinite* reducible (in the sense of BV) and hence requires in general the introduction of an infinite number of ghosts. Other examples of theories with this property are super p -branes [16] and string field theory (for a recent review see ref. [17]). The BRST quantization of string field theory was performed by Bocchichio [18] and Thorn [17]. The quantization of the Green–Schwarz superstring in the framework of BV has been presented only recently [1–3].

The main purpose of this paper is to perform the BRST_1 quantization of the Green-Schwarz superstring and in particular to give explicitly a rather simple form of the BRST_1 transformation rules in terms of the variables which occur in the quadratic action. We will discuss *under which conditions* the gauge fixed action can be shown to be BRST_1 invariant and we will emphasize the unusual features that occur when dealing with a system that involves an infinite number of fields.

The application of the BRST_1 procedure will enable us to construct in a relatively simple way the gauge fixed action in a wide class of gauges. In particular, we will prove the equivalence of the unitary spinorial light-cone gauge to covariant gauges. By making a number of on-shell trivial transformations we will also simplify the form of the BRST_1 transformation rules further. A particular role seems to be played by some spinorial on-shell conserved current. The relation between this spinorial current and that of the generators of the Virasoro algebra is still to be understood. We will also show that despite the complicated non-closure properties of the BRST_1 algebra, for the right-moving modes and for the left-moving modes the BRST_1 transformations are *off-shell* nilpotent.

As a spin-off of this paper we have found that unusual features occur when defining symmetries of a system that involves an infinite number of fields. Some properties of symmetries which are automatically fulfilled for systems with a finite number of fields, are not always satisfied in the case of infinite systems. This leads to non-trivial integrability conditions for symmetries of infinite systems. In particular, we will find that the requirement that the commutator of two symmetries gives another symmetry is non-trivial in the case of infinite systems. We will give explicit examples of models where this integrability condition is not satisfied. We find that in the case of the global BRST_1 symmetry of the GS gauge fixed action all integrability conditions are satisfied.

This paper is organized as follows. In sect. 2 we describe the BRST quantization of systems with an off-shell closed algebra, both irreducible and reducible ones. In sect. 3 the one-step Noether corrected procedure of quantization (BRST_1) of theories with an on-shell closed algebra is described. We present explicitly the difference between the standard BRST procedure and more general situations in which more than one-step Noether corrections may be required. In sect. 4 the procedure, described above, is applied to the heterotic string in the Green-Schwarz formulation. The gauge fixed action and the BRST_1 transformation rules are given in a class of gauges including the covariant gauge, which is quadratic in fields, as well as in a spinorial light-cone gauge. Sect. 5 deals with the BRST_1 symmetry in the free conformal gauge. In sect. 6 we discuss the subtleties that one encounters when dealing with symmetries of infinite systems, in particular the ones which are related to the fact that definitions of infinite sums are involved. We will also discuss some non-trivial integrability conditions for symmetries of infinite systems which are non-trivial to satisfy. We will show that for the BRST_1 global symmetry these integrability conditions are satisfied.

Finally, in appendix A we clarify some further issues of global and gauge symmetries of infinite systems. As examples we will discuss the superparticle, a toy model and the GS superstring.

2. The standard BRST formalism

The standard BRST procedure of quantization [7], which we will describe here can be applied to quantize all gauge theories which have an off-shell closed gauge algebra, both the irreducible as well as the (finite or infinite) reducible ones. Our notation will be such that we have a simple distinction between theories with off-shell closed algebras and standard BRST transformation rules and theories with on-shell closed algebras which require Noether-type corrections both to the gauge fixed action and the BRST transformation rules.

We generically denote the fields of the theory by φ_{cl} and the classical lagrangian by \mathcal{L}_{cl} . The gauge fixed action for theories with an irreducible or reducible algebra is constructed from set of fields which form the following pyramids [6, 9]:

$$\begin{array}{ccccccc}
 & & \varphi_{\text{cl}} & & & & \\
 & & \cdot & & c^1 & & \cdot \\
 & \cdot & & \cdot & & \cdot & \cdot \\
 & \cdot & \searrow & \cdot & \searrow & \cdot & \searrow \\
 \varphi_2 & \bar{\varphi}^1 & & \varphi_1 & \searrow & \bar{\varphi}^0 & \searrow \\
 & & & & \varphi_0 & & \mu_2 & \bar{\mu}^1 & \mu_1 & \bar{\mu}^0
 \end{array}
 \quad (2.1)$$

The top of the left pyramid in (2.1) contains the classical fields φ_{cl} . The first branch φ_0 contains the classical fields, the ghosts, the ghosts for ghosts etc. The branches $\bar{\varphi} = \{\bar{\varphi}^0, \bar{\varphi}^1, \dots\}$ contain all antighosts and the branches $\varphi_{\text{extra}} = \{\varphi_1, \varphi_2, \dots\}$ contain all extra ghosts. Thus the full left pyramid consists of $\varphi = \{\varphi_0, \varphi_{\text{extra}}\}$ and $\bar{\varphi}$. The right pyramid in (2.1) contains a special set of fields which will play the role of Lagrange multipliers imposing the gauge conditions. We denote the Lagrange multipliers corresponding to the antighosts by $\bar{\mu} = \{\bar{\mu}^0, \bar{\mu}^1, \dots\}$ and the Lagrange multipliers corresponding to the extra ghosts by $\mu = \{\mu_1, \mu_2, \dots\}$. The set of all fields contained in both pyramids is denoted by Φ .

The gauge fixed action for theories with an off-shell closed algebra has the following form:

$$\mathcal{L}_{\text{gf}} = \mathcal{L}_{\text{cl}} + s\Psi(\varphi, \bar{\varphi}), \quad (2.2)$$

where Ψ is the gauge fixing fermion, and

$$s\Phi^A = S^A(\Phi). \quad (2.3)$$

In general the $S^A(\Phi)$ are explicitly known for the classical fields φ_{cl} . They are given by the classical transformation rules with the parameter replaced by the first generation ghosts c^1 . The ones corresponding to the ghosts c^1, c^2, \dots are defined by the requirement that the BRST operator is nilpotent (see below). The main content of eq. (2.3) is the statement that the standard BRST transformation rules of all fields are gauge independent, i.e. do not depend on the gauge fermion Ψ .

The requirement that the gauge fixed action is BRST invariant leads to the condition

$$s\mathcal{L}_{\text{gf}} = s\mathcal{L}_{\text{cl}} + s^2\Psi(\Phi) = 0, \quad (2.4)$$

which is equivalent to the requirements that the classical lagrangian is gauge invariant,

$$s\mathcal{L}_{\text{cl}} = 0, \quad (2.5)$$

and that the BRST algebra closes off-shell,

$$s^2\Phi = 0. \quad (2.6)$$

Writing out eq. (2.6) in more detail leads to

$$s\varphi_0^A = S^A(\varphi_0), \quad s\varphi_{\text{extra}} = \mu, \quad s\mu = 0, \quad s\bar{\varphi} = \bar{\mu}, \quad s\bar{\mu} = 0. \quad (2.7)$$

Using this we deduce that the second condition (2.6) is fulfilled by construction on φ_{extra} and $\bar{\varphi}$. The non-trivial property of the given gauge theory is expressed by the statement that this condition also holds on φ_0 ,

$$s^2\varphi_0^A = sS^A(\varphi_0) = 0. \quad (2.8)$$

We can take the gauge fermion Ψ to be of the form

$$\Psi = \bar{\varphi}\chi(\varphi). \quad (2.9)$$

The gauge fixed action (2.2) then takes the form

$$\mathcal{L}_{\text{gf}} = \mathcal{L}_{\text{cl}} + \bar{\mu}\chi + \bar{\varphi}\frac{\partial\chi}{\partial\varphi_{\text{extra}}}\mu + \bar{\varphi}\frac{\partial\chi}{\partial\varphi_0^A}S^A(\varphi_0). \quad (2.10)$$

The BRST invariance of the gauge fixed action allows us to perform the following change of integration variables in the functional integral \mathcal{W}_{Ψ} :

$$\Phi^A \rightarrow \Phi'^A = S^A(\Phi) \cdot \delta\Psi. \quad (2.11)$$

In this way one can prove the gauge independence of the theory, i.e. $\mathcal{W}_{\Psi} = \mathcal{W}_{\Psi+\delta\Psi}$.

We note that the gauge fixed action (2.10) is linear in the antighosts. In reducible theories the dependence of the ghosts can be more complicated. In the case of gauge theories with an irreducible gauge algebra the action (2.10) reduces to the well-known Faddeev–Popov action. In the case of p -forms the action (2.10) has already been given in ref. [9].

3. The BRST₁ procedure

There exist gauge theories for which the off-shell closed algebra property expressed by eq. (2.6) is not valid and which therefore cannot be quantized using the standard BRST procedure. Examples of such theories are the non-abelian antisymmetric tensor model of Freedman and Townsend [13], the Green–Schwarz superstring [19] (or general super p -branes [16]) and string field theory [17]. In particular we have

$$s^2\varphi_0^A = S^{Ai}(\Phi) \frac{\partial S_{\text{cl}}}{\partial \varphi_{\text{cl}}^i}, \quad (3.1)$$

where S^{Ai} are the non-closure functions. From eq. (3.1) it follows that the gauge fixed action (2.2) is no longer invariant under the standard BRST transformation rules (2.7). However we see that eq. (2.6) is only violated by terms which are proportional to classical equations of motion. This gives us the possibility to restore the BRST invariance of the gauge fixed action by applying a standard Noether procedure. We first cancel the new terms in the variation of the action by adding new terms linear in the gauge fermion to the BRST transformation rules of the classical fields. These modifications we denote by Δs . Of course the substitution of these new variations into the part of the gauge fixed action not involving \mathcal{L}_{cl} will lead to new variations which are of higher order in the gauge fermion. According to the Noether procedure these terms have to be cancelled by adding new terms to the action which are of higher order in the gauge fermion and (possibly) new terms to the transformation rules, etc. It was shown in ref. [10] that in the case of supergravity this Noether procedure stops after one step, i.e. one ends up with a gauge fixed action that is quadratic in the gauge fermion and with modified BRST transformations that are at most linear in the gauge fermion. It turns out that the same is true for the non-abelian antisymmetric tensor, some of the p -branes, including the Green–Schwarz heterotic string, and string field theory. For all these theories one can show that the resulting gauge fixed action is given by

$$\begin{aligned} \mathcal{L}_{\text{gf}} &= \mathcal{L}_{\text{cl}} + \left(s + \frac{1}{2}\Delta s\right)\Psi \\ &= \mathcal{L}_{\text{cl}} + \bar{\mu}\chi + \bar{\varphi} \frac{\partial \chi}{\partial \varphi_{\text{extra}}} \mu + \bar{\varphi} \frac{\partial \chi}{\partial \varphi_0} \left(s + \frac{1}{2}\Delta s\right)\varphi_0 \\ &= \mathcal{L}_{\text{cl}} + \bar{\mu}\chi + \bar{\varphi} \frac{\partial \chi}{\partial \varphi_{\text{extra}}} \mu + \bar{\varphi} \frac{\partial \chi}{\partial \varphi_0^A} S^A + \frac{1}{2}\bar{\varphi} \frac{\partial \chi}{\partial \varphi_0^A} \bar{\varphi} \frac{\partial \chi}{\partial \varphi_0^B} S^{AB}. \end{aligned} \quad (3.2)$$

The s transformations are the standard ones defined in eq. (2.7). The extra Δs transformations are parametrized by means of the non-closure functions S^{AB} whose symmetry properties are such that the last term in eq. (3.2) is non-vanishing,

$$\Delta s \varphi_0^A = \frac{\partial \Psi}{\partial \Phi^B} S^{BA}. \quad (3.3)$$

The gauge fixed action (3.2) is invariant under the extended BRST_1 transformations. On φ_0^A they are given by

$$s_1 \varphi_0^A \equiv (s + \Delta s) \varphi_0^A = S^A + \frac{\partial \Psi}{\partial \Phi^B} S^{BA}. \quad (3.4)$$

On all other fields the BRST transformations are unchanged, i.e. $s_1 = s$. These extended BRST_1 transformations are nilpotent only *on-shell*, i.e. modulo the equations of motion corresponding to the gauge fixed action. Therefore on all fields we have

$$s_1^2 \Phi^A = \frac{dS_{\text{gf}}}{d\Phi^B} S^{BA}(\Phi). \quad (3.5)$$

Note that the coefficient $S^{BA}(\Phi)$ multiplying the field equation on the right-hand side of eq. (3.5) does not depend on the gauge fermion Ψ . This is a particular property of theories for which the BRST_1 procedure works, i.e. theories for which it is sufficient to extend the standard BRST gauge fixed action and transformation rules by just one step.

To show that the gauge fixed lagrangian (3.2) is invariant under the BRST_1 transformations s_1 , it is convenient to consider the gauge fixed action as a function of Φ and $\partial \Psi / \partial \Phi$,

$$\mathcal{L}_{\text{gf}} \left(\Phi, \frac{\partial \Psi}{\partial \Phi} \right) = \mathcal{L}_{\text{cl}} + \frac{\partial \Psi}{\partial \Phi^A} S^A + \frac{1}{2} \frac{\partial \Psi}{\partial \Phi^A} \frac{\partial \Psi}{\partial \Phi^B} S^{AB}. \quad (3.6)$$

In terms of \mathcal{L}_{gf} as a function of Φ and $\partial \Psi / \partial \Phi$ the BRST_1 transformations of Φ^A are given by

$$s_1 \Phi^A = \frac{\partial S_{\text{gf}}}{\partial (\partial \Psi / \partial \Phi^A)}. \quad (3.7)$$

From eq. (3.7) it follows that

$$s_1^2 \Phi^A = \frac{d\mathcal{L}_{\text{gf}}}{d(\partial \Psi / \partial \Phi^A)} \frac{\partial S_{\text{gf}}}{\partial (\partial \Psi / \partial \Phi^B)}. \quad (3.8)$$

Since on the other hand $s_1^2 \Phi^A$ is given by eq. (3.5) it follows that

$$\frac{\partial}{\partial(\partial\Psi/\partial\Phi^A)} \left\{ \frac{dS_{\text{gf}}}{d\Phi^B} \frac{\partial S_{\text{gf}}}{\partial(\partial\Psi/\partial\Phi^B)} \right\} = 0. \quad (3.9)$$

This equation tells us that all Ψ dependent terms in the BRST_1 variation of the gauge fixed action vanish. The Ψ independent part vanishes by itself due to the gauge invariance of the classical action: $sS_{\text{cl}} = 0$. Therefore the gauge fixed action (3.2) is invariant under the BRST_1 transformations s_1 .

In supergravity the particular form of the non-closure functions S^{AB} is related to the non-closure functions of the classical gauge algebra and to the fact that the Jacobi identities corresponding to the classical gauge algebra are only valid on-shell. In the case of the non-abelian antisymmetric tensor the S^{AB} are only related to the non-closure functions of the classical gauge algebra. The same applies to string field theory. Finally in the case of the Green–Schwarz superstring there are three types of non-trivial non-closure functions S^{AB} which have been described in ref. [15].

We thus see that although the interpretation of the functions S^{AB} in terms of the original gauge algebra is quite complicated (some of them are non-closure functions of the classical gauge algebra, others are non-closure functions of the Jacobi identities, etc.), their interpretation in terms of the BRST algebra is quite simple: all the S^{AB} are non-closure functions of the $s_1^2 = 0$ algebra.

Summarizing, we see that a characteristic feature that occurs in the quantization of theories with on-shell closed algebras (see eq. (3.5)) is that the BRST transformation rules receive one-step Noether-type corrections, i.e. they become extended BRST_1 transformations, and that the gauge-fixed action contains higher-order 3- and 4-ghost couplings, which are given by the last two terms in eq. (3.2).

The reason that in all known gauge theories, which were already quantized, at most a one-step Noether correction was sufficient, may be explained by the example of the Green–Schwarz superstring. From the general BV formalism [12] we learn that in general the gauge fixed action is of arbitrary order in the gauge fermion containing the higher-order non-closure functions S^{ABC} , S^{ABCD} , etc.,

$$S_{\text{gf}} = S_{\text{cl}} + \frac{\partial\Psi}{\partial\Phi^A} S^A + \frac{1}{2} \frac{\partial\Psi}{\partial\Phi^A} \frac{\partial\Psi}{\partial\Phi^B} S^{AB} + \frac{1}{3} \frac{\partial\Psi}{\partial\Phi^A} \frac{\partial\Psi}{\partial\Phi^B} \frac{\partial\Psi}{\partial\Phi^C} S^{ABC} + \dots \quad (3.10)$$

The requirement that the gauge fixed action should be BRST invariant leads in each order of the gauge fermion to an equation relating the S^A , S^{AB} , S^{ABC} , ... These equations define the higher-order non-closure functions. The first few equations are given by

$$S^A{}_{,B} S^B = S^{Ai} S^{\text{cl}}_{,i}, \quad (3.11)$$

$$\frac{1}{2} S^{AB}{}_{,C} + S^{(A}{}_{,D} S^{DB)} = S^{ABi} S^{\text{cl}}_{,i}, \quad (3.12)$$

$$\frac{1}{3} S^{ABC}{}_{,D} S^D + \frac{1}{2} S^{(AB}{}_{,D} S^{DC)} + S^{(A}{}_{,D} S^{DBC)} = S^{ABCi} S^{\text{cl}}_{,i}, \text{ etc.} \quad (3.13)$$

The content of eq. (3.5) which defines theories that can be quantized by means of the BRST_1 procedure is equivalent to that of eqs. (3.11)–(3.13) with all higher-order structure functions $S^{A_1 \dots A_n}$ ($n \geq 3$) set equal to zero.

From eqs. (3.11)–(3.13) it is easy to understand why the GS superstring can be quantized by means of the BRST_1 procedure just by counting derivatives. All terms in the GS action contain two derivatives and all terms in the κ -transformations contain one derivative. From eq. (3.11) it then follows that the functions S^{AB} do not contain any derivatives. It then follows from eqs. (3.12) and (3.13) that the function S^{ABC} and S^{ABCD} , in order to be non-zero, should contain negative powers of derivatives, which is clearly not possible. Therefore these functions have to vanish identically. Similar arguments apply to supergravity and the non-abelian antisymmetric tensor theory.

The only known theories for which the above analysis does not immediately apply are super p -branes ($p \geq 2$). We expect that these theories cannot be quantized by means of the BRST_1 procedure but have not worked out the higher-order structure functions S^{ABC} , S^{ABCD} etc. A more detailed analysis is necessary but will not be given here.

4. The BRST_1 quantization of the Green-Schwarz superstring

We now apply the BRST_1 procedure which was outlined in sect. 3 to quantize the heterotic Green-Schwarz superstring. The classical lagrangian involves the ten-dimensional coordinates (X^μ, θ) , the zweibeins $(e_z^a, e_{\bar{z}}^a)$ and the left-handed chiral fermions Ψ^I and is given by

$$\mathcal{L}(\text{heterotic}) = e \left\{ \Pi_z^\mu \Pi_{\bar{z}}^\mu + i \partial_z X^\mu \bar{\theta} \gamma_\mu \partial_{\bar{z}} \theta - i \partial_{\bar{z}} X^\mu \bar{\theta} \gamma_\mu \partial_z \theta - \frac{1}{2} i \Psi^I \partial_{\bar{z}} \Psi^I \right\}. \quad (4.1)$$

where $\Pi_z^\mu \equiv \partial_z X^\mu - i \bar{\theta} \gamma^\mu \partial_z \theta$ and $\Pi_{\bar{z}}^\mu \equiv \partial_{\bar{z}} X^\mu - i \bar{\theta} \gamma^\mu \partial_{\bar{z}} \theta$. To quantize the theory we use the set of fields which was described in ref. [1]. Using the notation of sect. 3 these fields are given by

$$\begin{aligned} \varphi_0 &= \{ X^\mu, e_z^a, e_{\bar{z}}^a, \Psi^I, c^a, \rho, \Lambda, \theta_{p,0} \}, & p \geq 0, & \quad \theta \equiv \theta_{0,0}, \\ \varphi_1 &= \{ \theta_{p,1} \}, \dots, \varphi_q = \{ \theta_{p,q} \}, & p \geq q, \\ \bar{\varphi}^0 &= \{ \bar{c}_a^z, \bar{c}_a^{\bar{z}}, \bar{\theta}_{\bar{z}}^{p,0} \}, \\ \bar{\varphi}^1 &= \{ \bar{\theta}_{\bar{z}}^{p,1} \}, \dots, \bar{\varphi}^q = \{ \bar{\theta}_{\bar{z}}^{p,q} \}, \\ \bar{\mu}^q &= \{ \bar{\mu}_a^z, \bar{\mu}_a^{\bar{z}}, \bar{\lambda}_{\bar{z}}^{p,q} \}, & q \geq 0, & \quad \mu_q = \{ \lambda_{p,q} \}, \quad q \geq 1. \end{aligned} \quad (4.2)$$

Here c_a , ρ and Λ are the ghosts corresponding to reparametrizations, conformal

transformations and Lorentz rotations. The $\{\theta_{p,0}\}$ are the infinite set of ghosts, ghosts for ghosts, etc. corresponding to the infinite reducible κ -transformations. The \bar{c}_a^z and $\bar{c}_a^{\bar{z}}$ are the antighosts associated to c^a , ρ and Λ while the $\{\bar{\theta}_z^{p,0}\}$ are the antighosts associated to $\{\theta_{p,0}\}$. Finally, the $\bar{\mu}_a^z$ and $\bar{\mu}_a^{\bar{z}}$ are the Lagrange multipliers for the reparametrizations, conformal transformations and Lorentz rotations, while the $\bar{\lambda}_z^{p,q}$ and $\lambda_{p,q}$ are the Lagrange multipliers for the infinite reducible κ -symmetry. Note that in our notation the index q labels the branch while the index p indicates the position of the field within a given branch.

The crucial observation of refs. [1–3] which allows us to construct a gauge fixed action corresponding to a free conformal field theory is the following. First of all, in a special gauge, which is a generalization of the gauge used in the quantization of the superparticle [4], one can use the possibility to rearrange the variables occurring in the general gauge fixed lagrangian (3.2) to avoid part of the functions $s_1\varphi_0$. In particular, one can get rid of the complicated functions $s_1\theta_{p,0}$, which have been calculated in ref. [3] and whose presence would lead to 4-ghost couplings of the form $\bar{\varphi}\varphi\varphi\varphi$. Secondly, one can choose a gauge such that the $\bar{\varphi}\varphi\varphi\varphi$ 4-ghost couplings corresponding to the last term in eq. (3.2) vanish.

We will show now that the above-mentioned rearrangement of variables leading to the absence of $\bar{\varphi}\varphi\varphi\varphi$ 4-ghost couplings can be performed for a wide class of gauges, which are characterized by the following gauge-fixing fermion:

$$\Psi = \bar{c}_z^z \chi_z^a + \bar{c}_a^{\bar{z}} \chi_{\bar{z}}^a + \sum_{q=0}^{\infty} \sum_{p=q}^{\infty} \bar{\theta}_z^{p,q} f \tilde{\theta}_{p,q}. \quad (4.3)$$

Here χ and f are arbitrary functions of X^μ , e_z^a , $e_{\bar{z}}^a$ and the $\tilde{\theta}$ variables are defined by

$$\tilde{\theta}_{p,q} \equiv \theta_{p,q} + \theta_{p+1,q+1} \quad \text{or} \quad \theta_{p,q} \equiv \sum_{r=0}^{\infty} (-1)^r \tilde{\theta}_{p+r,q+r}. \quad (4.4)$$

The second expression in this equation should be considered as a regularization for the infinite alternating sum. In sect. 5 we will see that this regularization is consistent with the BRST₁ symmetry of the gauge fixed action. In appendix A we will show that in general the consistency of the regularization provides a non-trivial integrability condition on the theory.

Simultaneously we also define

$$\pi_{p,0} \equiv s_1\theta_{p,0} + \lambda_{p+1,1}, \quad \pi_{p,q} \equiv \lambda_{p,q} + \lambda_{p+1,q+1}, \quad (q \geq 1). \quad (4.5)$$

The above redefinition of variables allows us to rewrite some of the terms in the

gauge fixed lagrangian as follows:

$$\begin{aligned}
 & \sum_{p=0}^{\infty} \frac{\partial \Psi}{\partial \theta_{p,0}} s_1 \theta_{p,0} + \sum_{q=1}^{\infty} \sum_{p=q}^{\infty} \frac{\partial \Psi}{\partial \theta_{p,q}} \lambda_{p,q} \\
 &= \bar{\theta}_z^{0,0} f_{s_1} \theta_{0,0} + (\bar{\theta}_z^{0,0} + \bar{\theta}_z^{1,1}) f \lambda_{1,1} + (\bar{\theta}_z^{1,1} + \bar{\theta}_z^{2,2}) f \lambda_{2,2} + \dots \\
 &+ \bar{\theta}_z^{1,0} f_{s_1} \theta_{1,0} + (\bar{\theta}_z^{1,0} + \bar{\theta}_z^{2,1}) f \lambda_{2,1} + (\bar{\theta}_z^{2,1} + \bar{\theta}_z^{3,2}) f \lambda_{3,2} + \dots \\
 &+ \dots \\
 &= \sum_{q=0}^{\infty} \sum_{p=q}^{\infty} \bar{\theta}_z^{p,q} f \pi_{p,q}. \tag{4.6}
 \end{aligned}$$

Note that in the above mechanism of redefining $s_1 \theta_{p,0}$ away it is essential that the summations in (4.6) are *infinite*.

We will now see that in the class of gauges described by eq. (4.3) the complicated functions $s_1 \theta_{p,0}$ not only do not occur in the gauge fixed action but are also absent in the BRST₁ transformation rules of the fields present in the lagrangian. Note that in order to describe the $\bar{\varphi} \bar{\varphi} \varphi \varphi$ 4-ghost couplings corresponding to the last term in (3.2) we do need *all* the non-closure functions η^{AB} and they have been described in ref. [10].

Using expression (3.2) for the gauge fixed BRST₁ lagrangian, the class of gauges specified by eq. (4.3) and the shift of variables (4.4) and (4.5) we obtain the following expression of \mathcal{L}_{gf} :

$$\begin{aligned}
 \mathcal{L}_{\text{gf}} = & \mathcal{L}_{\text{cl}} + \bar{\mu}_a^z \chi_z^a + \bar{\mu}_{\bar{a}}^{\bar{z}} \chi_{\bar{z}}^{\bar{a}} + \sum_{q=0}^{\infty} \sum_{p=q}^{\infty} \left\{ \bar{\theta}_z^{p,q} f \pi_{p,q} + \bar{\lambda}_{\bar{z}}^{p,q} f \tilde{\theta}_{p,q} \right\} \\
 & + \frac{\partial \Psi}{\partial X^\mu} (c^a \partial_a X^\mu - i \bar{\theta} \gamma^\mu \Pi_{\bar{z}} \theta_{1,0}) + \frac{\partial \Psi}{\partial e_z^a} (c^b \partial_b e_z^a + \partial_{\bar{z}} c^a - (\rho + \Lambda) e_z^a) \\
 & + \frac{\partial \Psi}{\partial e_z^a} (c^b \partial_b e_z^a - \partial_z c^a - (\rho - \Lambda) e_z^a + 4i \bar{\theta}_{1,0} \partial_z \theta) \\
 & + \frac{\partial \Psi}{\partial e_z^a} \frac{\partial \Psi}{\partial X^\mu} e_z^a (i \bar{\theta} \gamma^\mu \theta_{2,0} - 2i \bar{\theta}_{1,0} \gamma^\mu \theta_{1,0}). \tag{4.7}
 \end{aligned}$$

The BRST₁ transformations of the fields are given by

$$\begin{aligned}
s_1 X^\mu &= c^a \partial_a X^\mu - i \bar{\theta}^\mu \mathbb{I} \bar{\theta}_z \theta_{1,0} + \frac{\partial \Psi}{\partial e_z^a} e_z^a (i \bar{\theta} \gamma^\mu \theta_{2,0} - i \bar{\theta}_{1,0}^\mu \theta_{1,0}), \\
s_1 e_z^a &= c^b \partial_b e_z^a - \partial_z c^a - (\rho - \Lambda) e_z^a + 4i \bar{\theta}_{1,0} \partial_z \theta + e_z^a \sum_{p=0}^{\infty} \frac{dS_{\text{gf}}}{d\pi_{p,0}} \theta_{p+2,0} \\
&\quad + \frac{\partial \Psi}{\partial X^\mu} e_z^a (i \bar{\theta} \gamma^\mu \theta_{2,0} - 2i \bar{\theta}_{1,0} \gamma^\mu \theta_{1,0}), \\
s_1 e_z^a &= c^b \partial_b e_z^a - \partial_z c^a - (\rho + \Lambda) e_z^a, \\
s_1 \Psi^I &= c^a \partial_a \Psi^I - \frac{1}{2} (\rho - \Lambda) \Psi^I, \\
s_1 c^a &= c^b \partial_b c^a + 2i \bar{\theta}_{1,0} \mathbb{I} \bar{\theta}_z \theta_{1,0} e_z^a - 4i \frac{\partial \Psi}{\partial e_z^b} e_z^b \bar{\theta}_{1,0} \theta_{2,0} e_z^a, \\
s_1 (\rho - \Lambda) &= c^a \partial_a (\rho - \Lambda) - 4ie^{-1} \partial_a (e e_z^a) \bar{\theta}_{1,0} \mathbb{I} \bar{\theta}_z \theta_{2,0} + 8ie^{-1} \partial_b (e e_z^b) \frac{\partial \Psi}{\partial e_z^a} e_z^a \bar{\theta}_{1,0} \theta_{2,0}, \\
s_1 (\rho + \Lambda) &= c^a \partial_a (\rho + \Lambda) - 4i \partial_z (\bar{\theta}_{1,0} \mathbb{I} \bar{\theta}_z \theta_{1,0}) + 4i \partial_z \left(\frac{\partial \Psi}{\partial e_z^a} e_z^a \bar{\theta}_{1,0} \theta_{2,0} \right), \\
s_1 \bar{c}_a^z &= \bar{\mu}_a^z, \quad s_1 \bar{c}_a^{\bar{z}} = \bar{\mu}_a^{\bar{z}}, \\
s_1 \tilde{\theta}_{p,q} &= \pi_{p,q}, \quad s_1 \bar{\theta}_z^{p,q} = -\bar{\lambda}_z^{p,q}, \\
s_1 \pi_{p,q} &= \frac{dS_{\text{gf}}}{de_z^a} e_z^a \theta_{p+2,0} \delta_{q,0}, \\
s_1 \bar{\mu}_a^z &= s_1 \bar{\mu}_a^{\bar{z}} = s_1 \bar{\lambda}_z^{p,q} = 0.
\end{aligned} \tag{4.8}$$

Note that the BRST₁ transformation of $\pi_{p,q}$ is rather unusual. This is due to the shift of variables we performed in eq. (4.5). In fact, using the general expression for s_1^2 given by eq. (3.5) it is easy to derive the s_1 transformation of $\pi_{p,q}$:

$$s_1 \pi_{p,q} = s_1^2 \theta_{p,0} = \frac{dS_{\text{gf}}}{de_z^a} e_z^a \eta^{e_z^a, \theta_{p,0}} = \frac{dS_{\text{gf}}}{de_z^a} e_z^a \theta_{p+2,0} \delta_{q,0}. \tag{4.9}$$

We furthermore note that the s_1 transformations of both e_z^a and $\pi_{p,0}$ contain terms which are proportional to equations of motion. In fact both terms together consti-

tute a so-called trivial on-shell symmetry of the gauge fixed lagrangian (4.7),

$$\begin{aligned}\delta(\text{trivial})e_z^a &= e_z^a \sum_{p=0}^{\infty} \frac{dS_{\text{gf}}}{d\pi_{p,0}} \theta_{p+2,0}, \\ \delta(\text{trivial})\pi_{p,0} &= \frac{dS_{\text{gf}}}{de_z^a} e_z^a \theta_{p+2,0}.\end{aligned}\quad (4.10)$$

We could therefore get rid of these equation of motion terms in the BRST_1 transformations by redefining s_1 as $\tilde{s}_1 \equiv s_1 + \delta(\text{trivial})$.

We see that the gauge fixed action (4.8) in gauges where Ψ depend on both X^μ and e_z^a contains $\bar{\varphi}\bar{\varphi}\varphi\varphi$ 4-ghost couplings. In supergravity such $\bar{\varphi}\bar{\varphi}\varphi\varphi$ 4-ghost couplings are present in standard gauges. In the case of the non-abelian antisymmetric tensor and string field theory the same terms lead to $\bar{\varphi}\bar{\varphi}\varphi$ 3-ghost couplings. The peculiar feature of the Green-Schwarz superstring is that it is possible to choose a gauge independent of X^μ such that the 4-ghost couplings $\bar{\varphi}\bar{\varphi}\varphi\varphi$ are absent.

The fact that the functional integral is independent of the choice of the functions χ and f follows from the BRST_1 invariance of the gauge fixed action as was explained in sect. 3. We can consider several interesting gauges. For instance the function χ which fixes the reparametrizations can either correspond to a conformal gauge or a harmonic gauge. The function f which corresponds to the fixing of the infinite reducible κ -symmetry gives rise to a covariant gauge if we choose $f = \nabla_z$ [1–3] or a unitary spinorial light-cone gauge if we choose $g = \gamma^+ \gamma^-$. If we take the conformal, spinorial light-cone gauge we obtain the gauge fixed action of ref. [20]. We can also reproduce the arbitrary level truncation procedure of quantizing the Green-Schwarz string, which was performed in ref. [15], by making an appropriate choice of the function f . In that case we use for all ghosts of level $L < n$ the covariant gauge $f \sim \nabla_z$ and for all ghosts of level $L \geq n$ the spinorial light-cone gauge $f \sim \gamma^+ \gamma^-$. This shows that the truncation procedure of ref. [15] is a possible gauge where all ghosts of level $L \geq n$ for some given n become non-propagating and in that sense are absent in the truncated theory.

5. BRST_1 quantization of the Green-Schwarz superstring in the free gauge

In this section we consider the covariant gauge of refs. [1–3] for which the gauge fixed lagrangian (4.7) reduces to a free conformal field theory,

$$\Psi(\text{free}) = \bar{c}_a^z (e_z^a - (e_z^a)^0) + \bar{c}_a^{\bar{z}} (e_{\bar{z}}^a - (e_{\bar{z}}^a)^0) + \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} \bar{\theta}_z^{p,q} \nabla_z \tilde{\theta}_{p,q}, \quad (5.1)$$

where $(e_z^a)^0, (e_{\bar{z}}^a)^0$ are arbitrary background zweibeins. Since $\Psi(\text{free})$ does not depend on X^μ all $\bar{\varphi}\bar{\varphi}\varphi\varphi$ 4-ghost couplings in eq. (4.7) vanish. Furthermore the fact that in the gauge (5.1) the function f in eq. (4.7) is given by $f = \nabla_z$ enables us to redefine away all terms in eq. (4.7) which are proportional to $\partial_z\theta$. It is convenient to parametrize these terms as $K_{\bar{z}}\partial_z\theta$ where $K_{\bar{z}}$ is given by

$$\begin{aligned} K_{\bar{z}} &\equiv -2ie\partial_{\bar{z}}X^\mu\bar{\theta}\gamma_\mu - e(\bar{\theta}\gamma_\mu\partial_{\bar{z}}\theta)\bar{\theta}\gamma_\mu + 4ieb^{zz}\bar{\theta}_{1,0} \\ &= K_{\bar{z}}^{\text{cl}}(X, \theta) + 4ieb^{zz}\bar{\theta}_{1,0}. \end{aligned} \quad (5.2)$$

Note that $K_{\bar{z}}^{\text{cl}}$ is the classical spinorial current

$$K_{\bar{z}}^{\text{cl}} = \frac{\partial \mathcal{L}_{\text{cl}}}{\partial \partial_z \theta}. \quad (5.3)$$

Here the b fields are defined by

$$b^{zz} \equiv e_z^a \frac{\partial \Psi}{\partial e_z^a}, \quad b^{\bar{z}\bar{z}} \equiv e_{\bar{z}}^a \frac{\partial \Psi}{\partial e_{\bar{z}}^a}, \quad b^{z\bar{z}} \equiv e_z^a \frac{\partial \Psi}{\partial e_{\bar{z}}^a}, \quad b^{\bar{z}z} \equiv e_{\bar{z}}^a \frac{\partial \Psi}{\partial e_z^a}. \quad (5.4)$$

All the $K_{\bar{z}}\partial_z\theta$ terms in the gauge fixed lagrangian can now be redefined away as follows:

$$\begin{aligned} K_{\bar{z}}\partial_z\theta + \sum_{q=0}^{\infty} \sum_{p=q}^{\infty} \bar{\lambda}_{\bar{z}}^{p,q} \nabla_z \tilde{\theta}_{p,q} &= \sum_{q=0}^{\infty} \sum_{p=q+1}^{\infty} \bar{\lambda}_{\bar{z}}^{p,q} \nabla_z \tilde{\theta}_{p,q} + (K_{\bar{z}} + \bar{\lambda}_{\bar{z}}^{0,0}) \partial_z\theta \\ &\quad + (\bar{\lambda}_{\bar{z}}^{0,0} + \lambda_{\bar{z}}^{1,1}) \nabla_z \theta_{1,1} + (\bar{\lambda}_{\bar{z}}^{1,1} + \lambda_{\bar{z}}^{2,2}) \nabla_z \theta_{2,2} + \dots \\ &= \sum_{q=0}^{\infty} \sum_{p=q}^{\infty} \tilde{\pi}_{\bar{z}}^{p,q} \nabla_z \tilde{\theta}_{p,q}. \end{aligned} \quad (5.5)$$

The new $\tilde{\pi}_{\bar{z}}^{p,q}$ variables are related to the old $\bar{\lambda}_{\bar{z}}^{p,q}$ variables by

$$\tilde{\pi}_{\bar{z}}^{p,q} \equiv \bar{\lambda}_{\bar{z}}^{p,q} + (-1)^q K_{\bar{z}} \delta^{p,q}. \quad (5.6)$$

The variable $\tilde{\pi}_{\bar{z}}^{0,0}$ is the spinorial current of the gauge fixed action, $\tilde{\pi}_{\bar{z}}^{0,0} = \partial \mathcal{L}_{\text{gf}} / \partial \partial_z \theta$, which consists of the classical part $K_{\bar{z}}^{\text{cl}}$ and of a ghost contribution. In terms of these new shifted variables the gauge fixed lagrangian becomes a free conformal field theory. In what follows we have made the redefinitions

$$(\rho - \Lambda) \rightarrow (\rho - \Lambda) - \nabla_z c_{\bar{z}}, \quad (\rho + \Lambda) \rightarrow (\rho + \Lambda) - \nabla_{\bar{z}} c_z. \quad (5.7)$$

As has been explained in the second reference of ref. [1] one can then set $b^{z\bar{z}} = b^{\bar{z}z} = \rho = \Lambda = 0$ by using the field equation of $\rho + \Lambda$, $\rho - \Lambda$, $b^{z\bar{z}}$ and $b^{\bar{z}z}$. We thus find the following simple expression for the gauge fixed lagrangian:

$$\begin{aligned} \mathcal{L}_{\text{gf}}(\text{free}) = & e \partial_z X^\mu \partial_{\bar{z}} X^\mu - e(i/2) \Psi^I \partial_{\bar{z}} \Psi^I + \sum_{q=0}^{\infty} \sum_{p=q}^{\infty} \left\{ \bar{\theta}_{\bar{z}}^{p,q} \nabla_z \pi_{p,q} + \tilde{\pi}_{\bar{z}}^{p,q} \nabla_z \tilde{\theta}_{p,q} \right\} \\ & - eb^{zz} \nabla_z c_z - eb^{\bar{z}\bar{z}} \nabla_{\bar{z}} c_{\bar{z}} + \bar{\mu}_a^z \left(e_z^a - (e_z^a)^0 \right) + \bar{\mu}_{\bar{a}}^{\bar{z}} \left(e_{\bar{z}}^{\bar{a}} - (e_{\bar{z}}^{\bar{a}})^0 \right). \end{aligned} \quad (5.8)$$

Furthermore the BRST₁ transformations (4.8) in the gauge (5.1) are given by

$$\begin{aligned} s_1 X^\mu &= c^a \Pi_a^\mu - i\bar{\theta} \gamma^\mu s_1 \theta - ib^{zz} \bar{\theta}_{1,0} \gamma^\mu \theta_{1,0}, \\ s_1 \Psi^I &= c^a \partial_a \Psi^I + \frac{1}{2} (\nabla_z c_{\bar{z}}) \Psi^I, \\ s_1 c_z &= +c_z \nabla_{\bar{z}} c_z + 4ic_{\bar{z}} \bar{\theta}_{1,0} \partial_z \theta + 2i\bar{\theta}_{1,0} \not{I}_{\bar{z}} \theta_{1,0} - 4ib^{zz} \bar{\theta}_{1,0} \theta_{2,0}, \\ s_1 c_{\bar{z}} &= +c_{\bar{z}} \nabla_z c_{\bar{z}}, \\ s_1 b^{zz} &= T_{z\bar{z}}, \\ s_1 b^{\bar{z}\bar{z}} &= T_{zz}, \\ s_1 \tilde{\theta}_{p,q} &= \pi_{p,q}, \\ s_1 \bar{\theta}_{\bar{z}}^{p,q} &= -\tilde{\pi}_{\bar{z}}^{p,q} + (-1)^q K_{\bar{z}} \delta^{p,q}, \\ s_1 \tilde{\pi}_{\bar{z}}^{p,q} &= (-1)^q \delta^{p,q} T_{\bar{z}}, \\ s_1 e_z^a &= s_1 e_{\bar{z}}^{\bar{a}} = s_1 \bar{\mu}_a^z = s_1 \bar{\mu}_{\bar{a}}^{\bar{z}} = s_1 \pi_{p,q} = 0, \end{aligned} \quad (5.9)$$

where the bosonic T_{zz} , $T_{\bar{z}\bar{z}}$ and the fermionic $T_{\bar{z}}$ are defined by

$$\begin{aligned} T_{\bar{z}\bar{z}} &= \Pi_{\bar{z}}^\mu \Pi_{\bar{z}}^\mu - 2b^{zz} \nabla_{\bar{z}} c_z - (\nabla_{\bar{z}} b^{zz}) c_z + 4ib^{zz} \bar{\theta}_{1,0} \partial_{\bar{z}} \theta, \\ T_{zz} &= \Pi_z^\mu \Pi_z^\mu - \frac{1}{2} i \Psi^I \nabla_z \Psi^I - 2b^{\bar{z}\bar{z}} \nabla_z c_{\bar{z}} - (\nabla_z b^{\bar{z}\bar{z}}) c_{\bar{z}}, \\ T_{\bar{z}} &= s_1 K_{\bar{z}} = s_1 (K_{\bar{z}}^{\text{cl}}(X, \theta)) + 4ie T_{\bar{z}\bar{z}} \theta_{1,0} \\ &\quad + 4ieb^{zz} \left\{ c^a \partial_a \theta_{1,0} - (\nabla_{\bar{z}} c_z) \theta_{1,0} + \not{I}_{\bar{z}} \theta_{2,0} \right. \\ &\quad \left. + 4i\theta_{1,0} (\bar{\theta}_{1,0} \partial_{\bar{z}} \theta) - i\gamma^\mu \partial_{\bar{z}} \theta (\bar{\theta}_{1,0} \gamma_\mu \theta_{1,0}) - b^{zz} \theta_{3,0} \right\}. \end{aligned} \quad (5.10)$$

In eqs. (5.9) and (5.10) we have used that

$$s_1 \theta = \sum_{r=0}^{\infty} (-1)^r \pi_{r,r} = c^a \partial_a \theta + \bar{I} \bar{\theta}_{1,0} - b^{zz} \theta_{2,0}. \quad (5.11)$$

Furthermore we have redefined s_1 with the on-shell trivial symmetry given in eq. (4.10) and the following ones:

$$\begin{aligned} \delta(\text{trivial}) e_z^a &= e_z^a \left\{ \frac{dS_{\text{gf}}}{db^{zz}} + \sum_{q=0}^{\infty} (-1)^q \bar{\theta}_{1,0} \frac{dS_{\text{gf}}}{d\tilde{\pi}_z^{q,q}} \right\}, \\ \delta(\text{trivial}) e_{\bar{z}}^a &= e_z^a \frac{dS_{\text{gf}}}{db^{\bar{z}\bar{z}}}, \\ \delta(\text{trivial}) b^{zz} &= \frac{dS_{\text{gf}}}{de_z^a} e_z^a, \\ \delta(\text{trivial}) b^{\bar{z}\bar{z}} &= \frac{dS_{\text{gf}}}{de_{\bar{z}}^a} e_z^a, \\ \delta(\text{trivial}) \tilde{\pi}_z^{q,q} &= \frac{dS_{\text{gf}}}{de_z^a} e_z^a \bar{\theta}_{1,0}. \end{aligned} \quad (5.12)$$

In sect. 6 we will discuss under which conditions the gauge fixed action (5.8) can be shown to be invariant under the BRST₁ transformation rules (5.9) and (5.10).

The BRST₁ symmetry, given in eqs. (5.9) and (5.10), is closed only on-shell. However, concerning closure, the BRST₁ transformations have the following interesting property: if one restricts oneself both in the fields as well as in the transformations of the fields to only the left-moving (right-moving) modes, then the corresponding transformations are *off-shell* nilpotent. In particular, for the left-moving (l.m.) modes we have

$$\begin{aligned} s_1^{\text{lm}} X^\mu &= c_{\bar{z}} \partial_z X^\mu, \\ s_1^{\text{lm}} \Psi^I &= c_{\bar{z}} \partial_z \Psi^I, \\ s_1^{\text{lm}} c_{\bar{z}} &= c_{\bar{z}} \nabla_z c_{\bar{z}}, \\ s_1^{\text{lm}} b^{\bar{z}\bar{z}} &= \partial_z X^\mu \partial_z X^\mu - 2b^{\bar{z}\bar{z}} \nabla_z c_{\bar{z}} - (\nabla_z b^{\bar{z}\bar{z}}) c_{\bar{z}} - \frac{1}{2} i \Psi^I \nabla_z \Psi^I \end{aligned} \quad (5.13)$$

and

$$(s_1^{\text{lm}})^2 \{ X^\mu, \Psi^I, c_z, b^{\bar{z}\bar{z}} \} = 0. \quad (5.14)$$

Similarly, the transformations for the right-movers (r.m.) are given by

$$\begin{aligned} s_1^{\text{rm}} X^\mu &= c_z \Pi_z^\mu - i \bar{\theta} \gamma^\mu s_1 \theta - i b^{z\bar{z}} \bar{\theta}_{1,0} \gamma^\mu \theta_{1,0}, \\ s_1^{\text{rm}} c_z &= c_z \nabla_{\bar{z}} c_z + 2i \bar{\theta}_{1,0} \Pi_{\bar{z}} \theta_{1,0} - 4i b^{z\bar{z}} \theta_{1,0} \theta_{2,0}, \\ s_1^{\text{rm}} b^{z\bar{z}} &= T_{\bar{z}\bar{z}}, \\ s_1^{\text{rm}} \tilde{\theta}_{p,q} &= \pi_{p,q}, \\ s_1^{\text{rm}} \bar{\theta}_{\bar{z}}^{p,q} &= -\tilde{\pi}_{\bar{z}}^{p,q} + (-1)^q \delta^{p,q} K_{\bar{z}}, \\ s_1^{\text{rm}} \tilde{\pi}_{\bar{z}}^{p,q} &= (-1)^q \delta^{p,q} T_{\bar{z}}^{\text{rm}}, \end{aligned} \quad (5.15)$$

where $T_{\bar{z}}^{\text{rm}} \equiv s_1^{\text{rm}} K_{\bar{z}}$. For these right-moving modes we have

$$(s_1^{\text{rm}})^2 \{ X^\mu, c_z, b^{z\bar{z}}, \tilde{\theta}_{p,q}, \bar{\theta}_{\bar{z}}^{p,q}, \tilde{\pi}_{\bar{z}}^{p,q} \} = 0. \quad (5.16)$$

Eqs. (5.14) and (5.16) show in particular that

$$s_1^{\text{lm}} T_{zz} = s_1^{\text{rm}} T_{\bar{z}\bar{z}} = s_1^{\text{rm}} T_{\bar{z}}^{\text{rm}} = 0. \quad (5.17)$$

Our results suggest that the spinorial on-shell conserved current $K_{\bar{z}}$ and its variation $T_{\bar{z}}$ play an important role in the quantized superstring, which is analogous to the role that the Virasoro operators T_{zz} and $T_{\bar{z}\bar{z}}$ play in the quantized bosonic string theory.

6. Symmetries of infinite systems

In sects. 2–5 we have dealt with systems exhibiting infinite reducible symmetries. In this section we would like to give a more careful analysis of how to define global as well as local symmetries for systems with an infinite number of fields.

In the case of a system with a finite number of fields it is well known that the following two properties of global or local symmetries of an action S always automatically hold: (i) the algebra of symmetries form an off-shell or on-shell closed algebra; (ii) after a change of variables one obtains an action in terms of the new variables which is again symmetric, the symmetry being expressed in terms of the new variables.

We will now discuss when a system with an infinite number of fields satisfies the properties (i) and (ii) above. In general these properties are *not satisfied*. In appendix A we will give explicit examples of infinite models with symmetries that do not satisfy the properties (i) and (ii), e.g. the commutator of two symmetries does *not* give another symmetry of the action. Here we will first discuss the general case.

Consider an action of the following form:

$$S = \sum_{i=0}^{\infty} S^{(i)}, \quad (6.1)$$

where each $S^{(i)}$ consists of a finite number of terms. We will define the variation δS of S as follows:

$$\delta S \equiv \sum_{i,j=0}^{\infty} \frac{\delta S^{(i)}}{\delta \phi^j} \delta \phi^j \equiv \sum_{i,j=0}^{\infty} S_{,j}^{(i)} \delta \phi^j. \quad (6.2)$$

Suppose we can show that the action S is invariant under some gauge or global transformation ϵ , i.e. $\delta(\epsilon)S = 0$. This immediately leads to the following integrability condition:

$$\begin{aligned} \delta(\epsilon_1) \delta(\epsilon_2) S - (1 \leftrightarrow 2) &= \delta(\epsilon_1) \left\{ \sum_{i,j=0}^{\infty} S_{,j}^{(i)} \delta(\epsilon_2) \phi^j \right\} - (1 \leftrightarrow 2) \\ &= \sum_{i,j=0}^{\infty} S_{,j}^{(i)} \delta(\epsilon_1) \delta(\epsilon_2) \phi^j + \left[\delta(\epsilon_2), \sum_{i,j=0}^{\infty} S_{,j}^{(i)} \delta(\epsilon_1) \phi^j \right. \\ &\quad \left. - (1 \leftrightarrow 2) \right]. \end{aligned} \quad (6.3)$$

The first term at the r.h.s. of eq. (6.3) is the only one which arises in a system with a finite number of fields. The second term at the r.h.s. which involves the commutator $[\delta, \Sigma^\infty]$ can only arise in a system with an infinite number of fields. Clearly two symmetries only form an (on-shell) closed algebra if this commutator is vanishing, i.e.

$$\left[\delta, \sum_{i=0}^{\infty} \right] = 0. \quad (6.4)$$

The other issue which is obvious for systems with a finite number of fields but needs a clarification for infinite systems is related to performing a change of variables. In the quantization of the Green–Schwarz superstring we have seen that it may occur that after some change of variables the variation of the action under some gauge symmetry may contain an alternating infinite sum in terms of the new

variables, i.e.

$$\delta S \sim \sum_{i=0}^{\infty} (-1)^i \varphi'^i. \quad (6.5)$$

These alternating infinite sums are, in principle, not well defined. In the quantization of the Green-Schwarz string, however, we have found that the requirement that after a change of variables the original symmetry (in the case of the GS superstring this is the global BRST₁ symmetry) should be preserved as a symmetry of the action in terms of the new variables, leads to a unique definition of the alternating sum. In other words, there exists an unambiguous expression for the regularized infinite sum that satisfies the requirement that after the change of variables the action is still symmetric. So we have

$$S(\varphi) = S'(\varphi'), \quad \delta S'(\varphi') \sim \left\{ \sum_{i=0}^{\infty} (-1)^i \varphi'^i - f'(\varphi') \right\},$$

$$\sum_{i=0}^{\infty} (-1)^i \varphi'^i = f'(\varphi') = f(\varphi). \quad (6.6)$$

Eq. (6.6) should be consistent with the property (6.4) and therefore the following integrability condition should be satisfied:

$$\sum_{i=0}^{\infty} (-1)^i \delta \varphi'^i = \delta f'(\varphi') = \delta f(\varphi). \quad (6.7)$$

In the case of the GS superstring considered in sects. 2–5 the infinite sum $\sum_{i=0}^{\infty} (-1)^i \delta \varphi'^i$ is well defined since only a finite number of terms in the infinite sum are different from zero. In this case, as different from other systems which we will present in appendix A, the left-hand side of eq. (6.7) equals the right-hand side of eq. (6.7). This is a crucial integrability condition of the requirements (6.4) and (6.6).

We will now show more explicitly how the above general considerations work in case we consider the Green-Schwarz superstring and the global BRST₁ symmetry (5.9) of the gauge fixed action (5.8). Infinite alternating sums enter into the variation of the lagrangian (5.8) if we vary $\bar{\theta}_z^{p,q}$ in the $\bar{\theta} \nabla \pi$ terms and $\tilde{\pi}_z^{p,q}$ in the $\tilde{\pi} \nabla \tilde{\theta}$ terms since $s_1 \bar{\theta}_z^{p,q} \sim (-1)^q K_z \delta^{p,q}$ and $s_1 \tilde{\pi}_z^{p,q} = (-1)^q \delta^{p,q} T_z$. We thus have

$$s_1 \mathcal{L}_{\text{gf}}(\text{free}) \sim K_z \nabla_z \left\{ \sum_{q=0}^{\infty} (-1)^q \pi_{q,q} - s_1 \theta \right\} + T_z \nabla_z \left\{ \sum_{q=0}^{\infty} (-1)^q \tilde{\theta}_{q,q} - \theta \right\}. \quad (6.8)$$

If we would have worked with the original variables not making any field redefinitions we would of course end up with a similar variation but now expressed in terms of the old variables.

In terms of the original variables there is no problem in defining the infinite alternating sums which occur in the variation of the gauge fixed lagrangian. As we have explained for the general case above a natural way to define the infinite alternating sum occurring in the variation after a change of variables is to require that the symmetries of the lagrangian should be preserved under the change of variables. This leads us to define the following regularizations:

$$\sum_{q=0}^{\infty} \tilde{\theta}_{q,q} \equiv \theta, \quad (6.9)$$

$$\sum_{q=0}^{\infty} (-1)^q \pi_{q,q} \equiv s_1 \theta = c^a \partial_a \theta + \Pi_{\bar{z}} \theta_{1,0} - b^{zz} \theta_{2,0}. \quad (6.10)$$

Eq. (6.9) can be understood as the definition of the old θ variables in terms of the new $\tilde{\theta}$ variables [cf. eq. (4.4)]. The regularization (6.10) is consistent in the sense that the s_1 symmetry operator commutes with the infinite summation. For this to be the case it is essential that we have $s_1^2 \theta = 0$. Note that we achieved this by making the on-shell trivial transformation given in eq. (4.10). To actually prove that $s_1^2 \theta = 0$ the explicit form of $s_1 \theta_{1,0}$ and $s_1 \theta_{2,0}$ is required. One could alternatively say that $s_1 \theta_{1,0}$ and $s_1 \theta_{2,0}$ are *defined* as the solutions of the equations $s_1^2 \theta = 0$. We thus see that the explicit forms of $s_1 \theta_{1,0}$ and $s_1 \theta_{2,0}$ are not needed in showing the BRST₁ invariance of the gauge fixed lagrangian in terms of the new variables. Rather they arise as the solutions to certain integrability conditions. The same thing happens in fact for all the other variations $s_1 \theta_{p,0}$. Their explicit forms are rather complicated functions of $\Pi_{\bar{z}}$, $\theta_{p,0}$ and b^{zz} and have been calculated in ref. [3]. However, these complicated functions do not occur in the action, neither do we need to know them in order to prove that the gauge fixed lagrangian (5.8) is BRST₁ invariant, at least for the general class of gauges (4.3) we have considered in this paper. Rather they do arise as the solutions to the set of integrability conditions $s_1^p \theta = 0$.

The definition of the alternative infinite sum of free variables in eq. (6.10) and the more general relation, which is required for integrability of the BRST₁ symmetry of the action (5.8),

$$\sum_{q=0}^{\infty} (-1)^q \pi_{q+p,q} \equiv s_1 \theta_{p,0}, \quad p = 0, \dots, \infty \quad (6.11)$$

forces us to ask the following question: Should we consider the fields $\pi_{p,q}$, in terms of which the action (5.8) is quadratic, as free fields or as constrained fields?

It is instructive to compare this situation with the chiral dynamics action $\frac{1}{2}(\partial_\mu \phi^I)^2$, where the ϕ^I satisfy the constraint

$$(\phi^I)^2 = 1, \quad I = 1, \dots, N. \quad (6.12)$$

Despite the fact that action in terms of the N fields ϕ^I is quadratic, it becomes the equivalent but highly nonlinear action

$$\frac{1}{2}(\partial_\mu \phi^i)^2 + \left(\partial_\mu \sqrt{1 + (\phi^i)^2} \right)^2, \quad i = 1, \dots, N-1, \quad (6.13)$$

when the constraint (6.14) is solved and the system depends on $N-1$ interacting unconstrained fields instead of N free constrained fields. Coming back to our example we see that asking the question: are the fields $\pi_{p,q}$ with $0 \leq q < \infty$, $q \leq p < \infty$ free fields or constrained fields, requires clarification. Contrary to the chiral dynamics case, eq. (6.11) does not allow us to express the infinite number of fields $\pi_{p,q}$ through some number of fields $\pi_{p,q}$, which is less than the original set. So in our interpretation eq. (6.11) should not be considered as a constraint but rather as a regularization of the infinite alternating sum. Accordingly, it is correct to say that the action (5.8) is a free field action.

If in chiral dynamics there would be no constraint (6.12) then the sum of the finite number of fields $(\phi^I)^2$ would be a definite, well-defined function of ϕ^I . In our case the infinite alternative sum $\sum_{q=0}^{\infty} (-1)^q \pi_{q+p,q}$ is not well defined a priori. Therefore, the fact that this sum must be defined to be equal to the right-hand side of eq. (6.11) is not in contradiction with the possibility to define the functional integration over the variables $\pi_{p,q}$ as an integration over free variables. The fact that the counting of physical degrees of freedom and of the conformal anomaly, using the lagrangian (5.8) as a free lagrangian, gives a correct result [1–3] confirms our belief that the situation is very different from that in chiral dynamics and should be further explored.

To conclude our discussion of symmetries of infinite systems we remark that the infinite set of spinorial ghosts which arise in the quantization of the GS superstring is related to the representation of an OSp algebra [21–24] with a finite graded dimension [21,23]. In particular, the regularized graded dimension \mathcal{D} of $\text{OSp}(2n-q, q|2m)$ is given by [21,23],

$$\mathcal{D} = \frac{2^{n-1}}{(1+x)^m}. \quad (6.14)$$

Since a counting of the number of degrees of freedom N of the superstring gives [1–3]

$$N = \frac{16}{(1+x)^2}, \quad (6.15)$$

this indicates that the relevant orthosymplectic supergroup describing the degrees of freedom of the superstring, as given in refs. [1–3], is given by $\text{OSp}(9, 1|4)$ [23]. The relation to orthosymplectic supergroups is less clear in the regularized expression \mathcal{A} for the conformal anomaly, as given in refs. [1–3],

$$\mathcal{A} = -32 \left(\frac{12}{(1+x)^4} - \frac{12}{(1+x)^3} + \frac{1}{(1+x)^2} \right). \quad (6.16)$$

This equation is still lacking a group theoretical explanation. The group theoretical origin of the above expression for the conformal anomaly will hopefully provide us with a further understanding of the underlying algebraic structure as well as with the correct mathematical framework in which to describe the BRST_1 symmetry of the particular infinite system considered in this paper.

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Note added

After the completion of this paper we received a preprint by Bastianelli et al. [26], who discuss the covariant gauge fixing of the superparticle. They propose to use only one branch (in the terminology of eq. (17) of ref. [1]) instead of an infinite number of branches. However, in order to obtain a correct number of degrees of freedom it is essential to have an infinite alternating sum over the branches [1–3]. Indeed, the gauge fixed action of ref. [26], although it avoids the need to regularize infinite alternating sums [see our discussion around eqs. (6.8)–(6.11)], does not produce the correct counting of degrees of freedom.

Appendix A

In this appendix we will consider the quantization of the *superparticle* in the covariant gauge as performed in ref. [4] and the issue of residual gauge invariances of the gauge fixed action, which has been discussed recently in the literature [25].

To illustrate some of the ideas concerning symmetries of infinite systems which we discussed in sect. 6, we also present in this appendix a simple *toy model*. The toy model will show us how easy it is to introduce a “symmetry” in a theory with an infinite number of fields and on the other hand it will teach us how non-trivial it is to do this in such a way that certain integrability conditions are satisfied.

Finally we will present gauge symmetries à la Fisch–Henneaux [25] of the gauge fixed action corresponding to the *GS Superstring*. We will show that unlike in the case of the superparticle the “gauge symmetries” do not satisfy the integrability conditions discussed above. Thus the Green–Schwarz gauge fixed action provides us with still another example of how non-trivial it is to have a gauge symmetry of an infinite system.

A.1. THE SUPERPARTICLE

The superparticle gauge fixed action has been given in ref. [4] and was discussed in ref. [25]. In ref. [25] the claim has been made that the superparticle gauge fixed action of ref. [4] possesses residual gauge symmetries. We will now show first that there is in fact an infinite number of gauge symmetries à la Fisch–Henneaux which are all of the so-called Stueckelberg type, i.e. they correspond to symmetries which express the fact that only certain fixed combinations of the variables occur in the gauge fixed action or in other words they express the fact that certain variables do not occur in the action. To see this we first rewrite the superparticle action, using the notation of refs. [4, 25] as follows:

$$\begin{aligned}
 S_1 = \int d\tau \bigg\{ & p_\mu \dot{x}^\mu - \frac{1}{2} g p^2 + \Pi_\xi (g - 1) + \hat{\chi} \dot{\chi} - \sum_{i=1}^{\infty} \dot{\bar{c}}_i^0 \Pi_{i+1}^{i+1} - \dot{\bar{\theta}} (\Pi_1^1 - \not{p} \theta + 4 c_1^0 \hat{\chi}) \\
 & - \sum_{i=1}^{\infty} \dot{\bar{c}}_i^i (\Pi_{i+1}^1 + \not{p} c_i^0 - 2 c_{i+2}^0 \hat{\chi}) - \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \dot{\bar{c}}_i^j (\Pi_{i-1}^{i-j} + \Pi_{i+1}^{i-j+1}) \bigg\} \\
 = \int d\tau \bigg\{ & - \frac{1}{2} g p^2 + \Pi_\xi (g - 1) - \dot{\bar{\theta}} (\Pi_1^1 - \not{p} \theta + 4 c_1^0 \hat{\chi}) \\
 & + p_\mu (x^\mu - \bar{c}_1^1 \gamma^\mu c_1^0 - \bar{c}_2^2 \gamma^\mu c_2^0 - \bar{c}_3^3 \gamma^\mu c_3^0 - \dots) + \hat{\chi} (\chi - 2 \bar{c}_1^1 c_2^0 - 2 \bar{c}_2^2 c_3^0 - 2 \bar{c}_3^3 c_4^0) \\
 & - \dot{\bar{c}}_1^0 (\Pi_2^2 - \not{p} c_1^1) - \dot{\bar{c}}_2^0 (\Pi_3^3 - \not{p} c_2^2 - 2 c_1^1 \hat{\chi}) \\
 & - \dot{\bar{c}}_3^0 (\Pi_4^4 - \not{p} c_3^3 - 2 c_2^2 \hat{\chi}) - \dot{\bar{c}}_4^0 (\Pi_5^5 - \not{p} c_4^4 - 2 c_3^3 \hat{\chi}) - \dots \\
 & - (\bar{c}_1^1 - \bar{c}_3^3) \Pi_2^1 - (\bar{c}_3^3 + \bar{c}_5^5) \Pi_4^2 - (\bar{c}_5^5 + \bar{c}_7^7) \Pi_6^3 - \dots \\
 & - \dot{\bar{c}}_3^1 (\Pi_2^2 + \Pi_4^4) - \dot{\bar{c}}_5^2 (\Pi_4^4 + \Pi_6^6) - \dot{\bar{c}}_7^3 (\Pi_6^6 + \Pi_8^8) - \dots \\
 & - (\bar{c}_2^2 + \bar{c}_4^4) \Pi_3^1 - (\bar{c}_4^4 + \bar{c}_6^6) \Pi_5^2 - (\bar{c}_6^6 + \bar{c}_8^8) \Pi_7^3 - \dots \\
 & - \dot{\bar{c}} (\Pi_3^3 + \Pi_5^5) - \dot{\bar{c}}_6^2 (\Pi_5^5 + \Pi_7^7) - \dot{\bar{c}}_8^3 (\Pi_7^7 + \Pi_9^9) - \dots \\
 & - (\bar{c}_3^3 + \bar{c}_5^5) \Pi_4^1 - (\bar{c}_5^5 + \bar{c}_7^7) \Pi_6^2 - (\bar{c}_7^7 + \bar{c}_9^9) \Pi_8^3 - \dots \\
 & - \dot{\bar{c}}_5^1 (\Pi_4^4 + \Pi_6^6) - \dot{\bar{c}}_7^2 (\Pi_6^6 + \Pi_8^8) - \dot{\bar{c}}_9^3 (\Pi_8^8 + \Pi_{10}^{10}) - \dots \bigg\}. \tag{A.1}
 \end{aligned}$$

Hence we see that the c_i^i ($i = 1, 2, \dots$) occur in the following combinations:

$$\begin{aligned} & (x^\mu - \bar{c}_1^1 \gamma^\mu c_1^0 - \bar{c}_2^2 \gamma^\mu c_2^0 - \dots), \\ & (\chi - 2\bar{c}_1^1 c_2^0 - 2\bar{c}_2^2 c_3^0 - 2\bar{c}_3^3 c_4^0 - \dots), \\ & (\Pi_2^2 - \not{p} c_1^1), \quad (\Pi_3^3 - \not{p} c_2^2 - 2c_1^1 \hat{\chi}), \quad (\Pi_4^4 - \not{p} c_3^3 - 2c_2^2 \hat{\chi}), \quad (\text{A.2}) \\ & \dots \text{etc.} \end{aligned}$$

Therefore the c_i^i ($i = 1, 2, \dots$) variables can be effectively redefined away from the action in the same way as this happens in the Stueckelberg mechanism. The fact that the c_i^i ($i = 1, 2, \dots$) variables effectively do not occur in the superparticle action means that there is an additional gauge symmetry which enables one to gauge these variables away. From the above one deduces that these gauge symmetries are given by

$$\begin{aligned} \delta x^\mu &= \bar{\epsilon}_j \gamma^\mu c_{j+1}^0, \\ \delta \chi &= 2\bar{\epsilon}_j c_{j+2}^0, \\ \delta \bar{c}_{1+2i+j}^{1+i+j} &= (-1)^i \bar{\epsilon}_j, \\ \delta \Pi_{2+2i+j}^{2+i+j} &= (-1)^i \not{p} \epsilon_j, \\ \delta \pi_{3+2i+j}^{3+i+j} &= (-1)^i 2\epsilon_j \hat{\chi}, \end{aligned} \quad (\text{A.3})$$

for $j = 0, 1, 2, \dots, *$.

The important point now is that in order to show that the superparticle action S_1 is invariant under the gauge symmetries given in (A.3) one needs to define infinite summations. For instance to show that S_1 is invariant under the gauge symmetry with parameter ϵ_1 , one needs to define the following infinite summations:

$$\begin{aligned} \Pi_2^1 &= - \sum_{i=1}^{\infty} (-1)^i (\Pi_{2i}^i + \Pi_{2i+2}^{i+1}), \\ \bar{c}_3^1 &= - \sum_{i=1}^{\infty} (-1)^i (\bar{c}_{1+2i}^i + \bar{c}_{3+2i}^{1+i}). \end{aligned} \quad (\text{A.4})$$

Since all the variables which occur in the above infinite summations are invariant under the ϵ_1 gauge symmetry we do not get any inconsistency.

*The $j = 1$ symmetry is the one given in ref. [25].

One might now wish to perform the change of variables described in refs. [4, 25]:

$$\begin{aligned}
 \tilde{\Pi}_1^1 &= \Pi_1^1 - \not{p}\theta + 4c_1^0\hat{\chi}, \\
 \tilde{\Pi}_{i+1}^{i+1} &= \Pi_{i+1}^{i+1}, \quad i \geq 1, \\
 \tilde{\Pi}_{i+1}^1 &= \Pi_{i+1}^1 + \not{p}c_i^0 - 2c_{i+1}^0\hat{\chi}, \quad i \geq 1, \\
 \tilde{\Pi}_{i+j+3}^{i+2} &= \Pi_{i+j+3}^{i+2} + \Pi_{i+j+1}^{i+1}, \quad i, j \geq 0, \\
 \tilde{\Pi}_\xi &= \Pi_\xi - \tfrac{1}{2}p^2.
 \end{aligned} \tag{A.5}$$

Note that this redefinition differs from the one given in (A.2) and one should not confuse the two. The difference between the two is that after the change of variables (A.2) the action can be written in terms of the new variables which are gauge inert but the action is not quadratic. On the other hand performing the change of variables (A.5) one ends up with an action in terms of the new variables which is quadratic but gauge invariant in a non-trivial way. The action S^2 in terms of the new variables (A.5) is given by

$$S_2 = \int d\tau \left\{ p_\mu \dot{x}^\mu - \tfrac{1}{2}p^2 + \tilde{\Pi}_\xi(g-1) + \hat{\chi}\dot{\chi} - \dot{\theta}\tilde{\Pi}_1^1 - \sum_{i=1}^{\infty} \sum_{j=0}^i \dot{c}_i^j \tilde{\Pi}_{i+1}^{i-j+1} \right\}. \tag{A.6}$$

This action is again invariant under the ϵ_1 gauge symmetry provided that one now defines the following infinite summation:

$$\sum_{i=0}^{\infty} (-1)^i \tilde{\Pi}_{2i+2}^{i+1} = \not{p}c_1^0 - 2c_2^0\hat{\chi}. \tag{A.7}$$

Note that again all the variables in the above infinite summation are inert under the ϵ_1 gauge symmetry and therefore this is a consistent regularization. Of course one would expect the action S_2 to be gauge invariant since the gauge invariance in this case just means that a certain variable does not occur in the action. This was the case in terms of the old variables under the condition (A.4) and remains to be true in terms of the new variables under the condition (A.7). It has been claimed in ref. [25] that only the action S_1 in terms of the old variables is gauge invariant but not the action S_2 in terms of the new variables. In our interpretation of symmetries of infinite systems this is not the case. Our statement is that both the action S_1 as well as the action S_2 is gauge invariant under the conditions (A.4) and (A.7) respectively. In both cases we have to deal with infinite alternating sums which a priori are not well defined and need to be regularized. By definition we regularize the infinite

sums in such a way that the gauge symmetry is preserved after the change of variables.

Our conclusion is that the gauge fixed superparticle action is consistently gauge invariant under an infinite number of gauge symmetries of the type given in ref. [25]. However all these gauge symmetries are trivial in the sense that all of them are of the Stueckelberg type. Thus the situation of the superparticle resembles the situation of massive vector fields, which can be presented in the form

$$\mathcal{L} = \frac{1}{2} F_{\mu\nu}^2(A) + \frac{1}{2} m^2 (\nabla_\mu \phi)^2, \quad (\text{A.8})$$

where $\nabla_\mu \phi = \partial_\mu \phi + m A_\mu$. Written in this form the action has the following gauge symmetry:

$$\delta A_\mu = \partial_\mu \xi, \quad \delta \phi = -\xi. \quad (\text{A.9})$$

However we may rewrite the action in terms of the new variable $W_\mu = A_\mu + \partial_\mu \phi$,

$$\mathcal{L} = \frac{1}{2} F_{\mu\nu}^2(W_\mu) + \frac{1}{2} m^2 W_\mu^2, \quad (\text{A.10})$$

i.e. in terms of variables which are gauge inert.

We finally note that the fact that the gauge fixed superparticle action depends on all fields c_j^i except the ones with $i=j$ does not affect the counting of degrees of freedom since effectively it makes the number of branches (note that $c_i^i \sim \bar{\theta}_z^{p,0}$ in our notation) less by one unit. Thus from the infinite number of alternating branches one branch is extracted. The result of the Euler regularized infinite summation is not changed by this.

A.2. A TOY MODEL

It is useful to illustrate the subtleties that are involved in dealing with symmetries of infinite systems by means of the following toy model. Consider the classical Green–Schwarz action,

$$\mathcal{L}(\text{GS}) = e \left\{ \Pi_z^\mu \Pi_{\bar{z}}^\mu + i \partial_z X^\mu \bar{\theta} \gamma_\mu \partial_{\bar{z}} \theta - i \partial_{\bar{z}} X^\mu \theta \gamma_\mu \partial_z \theta \right\}, \quad (\text{A.11})$$

and the following variations of X^μ and θ :

$$\delta X^\mu = \bar{\epsilon} \gamma^\mu \theta, \quad \delta \theta = \epsilon. \quad (\text{A.12})$$

Note that for $\epsilon = I_{\bar{z}}^\mu \kappa_z$ these transformations coincide with a κ transformation of X^μ and θ . The Green–Schwarz action is not invariant under these variations,

$$\delta(\epsilon) \mathcal{L}(\text{GS}) = -4ie \bar{\epsilon} I_{\bar{z}}^\mu \partial_z \theta. \quad (\text{A.13})$$

Note that in the case of κ transformations $\epsilon = \mathbb{I}\!\!\mathbb{I}_{\bar{z}} \kappa_z$ with the κ variation (A.13) can be cancelled by a κ transformation of the zweibein;

$$\delta(\kappa) e_z^a = 4i(\bar{\kappa}_z \partial_z \theta) e_z^a. \quad (\text{A.14})$$

Instead of doing this we can also cancel the ϵ variation (A.13) by introducing a Lagrange multiplier $\bar{\lambda}_z^{0,0}$ with the transformation rule

$$\delta \lambda_z^{0,0} = \mathbb{I}\!\!\mathbb{I}_{\bar{z}} \epsilon. \quad (\text{A.15})$$

The variation (A.13) can then be cancelled by adding to the Green-Schwarz action the term

$$\bar{\lambda}_z^{0,0} \partial_z \theta. \quad (\text{A.16})$$

Of course the addition of this new term to the Green-Schwarz action does not restore the full ϵ invariance since we also have to vary θ in (A.16). This leads to an additional variation which can be cancelled by introducing yet another Lagrange multiplier $\theta_{1,1}$ with the transformation law $\delta \theta_{1,1} = -\epsilon$ and by adding to the action the term $\bar{\lambda}_z^{0,0} \partial_z \theta_{1,1}$. This term in turn leads to a new variation whose cancellation requires the introduction of a Lagrange multiplier $\bar{\lambda}_z^{1,1}$ etc.

It is by now clear that in order to obtain an ϵ invariant action we are forced to introduce an infinite number of Lagrange multipliers $\bar{\lambda}_z^{p,p}$ ($p = 0, 1, 2, \dots$) and $\theta_{q,q}$ ($q = 1, 2, \dots$). We thus obtain the following toy model (TM) action:

$$\begin{aligned} \mathcal{L}(\text{TM}) &= \mathcal{L}(\text{GS}) + \bar{\lambda}_z^{0,0} \partial_z \theta + \bar{\lambda}_z^{0,0} \partial_z \theta_{1,1} + \bar{\lambda}_z^{1,1} \partial_z \theta_{1,1} + \bar{\lambda}_z^{1,1} \partial_z \theta_{2,2} + \dots \\ &= \mathcal{L}(\text{GS}) + \sum_{p=0}^{\infty} \bar{\lambda}_z^{p,p} \partial_z \tilde{\theta}_{p,p}, \end{aligned} \quad (\text{A.17})$$

where the $\tilde{\theta}$ variables are defined by

$$\tilde{\theta}_{p,p} \equiv \theta_{p,p} + \theta_{p+1,p+1}, \quad p = 0, 1, 2, \dots; \quad \theta \equiv \theta_{0,0}. \quad (\text{A.18})$$

The ϵ transformations of the fields are given by

$$\delta X^\mu = \bar{\epsilon} \gamma^\mu \theta, \quad \delta \theta_{p,p} = (-1)^p \epsilon, \quad \delta \bar{\lambda}_z^{p,p} = (-1)^p \bar{\epsilon} \mathbb{I}\!\!\mathbb{I}_{\bar{z}}. \quad (\text{A.19})$$

In which sense have our naive multiplications described above led us to a truly gauge invariant action? In order to show that the toy model action (A.18) is invariant under the ϵ transformations given by (A.19) we need to define the

following infinite summation:

$$\sum_{p=0}^{\infty} (-1)^p \tilde{\theta}_{p,p} = \theta. \quad (\text{A.20})$$

In contrast to the superparticle model described above this infinite summation involves variables that do transform under the gauge transformations. The consistency of the regularization (A.20) which was needed for the proof of the gauge invariance of the toy model action requires that the gauge variation of eq. (A.20) is well defined. In particular it means that the gauge variation $\delta(\epsilon)$ commutes with the infinite summation Σ in eq. (A.20). Unfortunately this is not the case. On the one hand varying the r.h.s. of eq. (A.20) we obtain ϵ . On the other hand, if we first vary all the terms on the l.h.s. of eq. (A.20) and then perform the infinite summation we obtain zero since all the $\tilde{\theta}$ variables are inert. Therefore it is incorrect to say that the toy model action (A.18) is gauge invariant under the transformations (A.19).

Another way to see that the toy model is not gauge invariant is to look at the commutator of two ϵ transformations. Usually the commutator of two symmetries is supposed to give another symmetry of the action. In our case we find that the commutator of two ϵ transformations is given by

$$\begin{aligned} [\delta(\epsilon_1), \delta(\epsilon_2)] X^\mu &= \xi^\mu, \\ [\delta(\epsilon_1), \delta(\epsilon_2)] \bar{\lambda}_{\bar{z}}^{p,p} &= (-1)^p \partial_{\bar{z}} \bar{\theta}^{\bar{z}}, \end{aligned} \quad (\text{A.21})$$

where $\xi^\mu = \bar{\epsilon}_2 \gamma^\mu \epsilon_1$. One can easily check that the bosonic ξ transformations are *not* a gauge symmetry of the toy model action.

A necessary consistency requirement of any gauge symmetry is that the commutator of two gauge symmetries yields another gauge symmetry. The fact that in the above toy model this was not the case is immediately related to the fact that the two operations variation (δ) and infinite summation (Σ) did not commute, i.e. $[\delta, \Sigma] \neq 0$.

Thus the above toy model has provided us with an example of an infinite system which at first sight seems to have a gauge symmetry. However this gauge symmetry does not satisfy certain integrability conditions. This means that we cannot use this gauge symmetry to eliminate for instance some degrees of freedom. It is in this sense that we would call the gauge symmetry (A.19) a false gauge symmetry.

A.3. THE GREEN-SCHWARZ SUPERSTRING

We will now show that the gauge fixed GS superstring action does not have any consistent gauge invariances of the type we encountered in the case of the superparticle [see eq. (A.3)]. Suppose we would try to generalize the $j=1$ gauge symmetry [25] of the superparticle action to the case of the superstring. An obvious generaliza-

tion is the following one:

$$\begin{aligned}
 \delta X^\mu &= -i\bar{\epsilon}_{\bar{z}}\gamma^\mu\theta_{1,0}, \\
 \delta c_z &= -4i\bar{\epsilon}_{\bar{z}}\theta_{2,0}, \\
 \delta\bar{\theta}_{\bar{z}}^{p,q} &= (-1)^q\delta^{p,q}\bar{\epsilon}_{\bar{z}}, \\
 \delta\bar{\lambda}_{\bar{z}}^{p,q} &= (-1)^q\left\{\delta^{p,q}\partial_{\bar{z}}\bar{\theta}_{\bar{z}}\gamma_\mu(\bar{\epsilon}_{\bar{z}}\gamma^\mu\theta_{1,0}) + \delta^{p,q+1}\bar{\epsilon}_{\bar{z}}\Pi_{\bar{z}} + \delta^{p,q+2}b^{z\bar{z}}\bar{\epsilon}_{\bar{z}}\right\}, \\
 \delta\lambda_{p+q+1,q+1} &= (-1)^q\delta(s_1\theta_{p,0}).
 \end{aligned} \tag{A.22}$$

The gauge fixed action does indeed satisfy $\delta(\epsilon_{\bar{z}})S = 0$. However, one can easily see that in this case the integrability condition (6.7) is not satisfied. The essential difference with the case of the superparticle is that the transformation rules themselves are not gauge invariant. Therefore $[\delta(\epsilon_{\bar{z}}^1), \delta(\epsilon_{\bar{z}}^2)] \neq 0$ and gives something which is not a symmetry of the action. Another way to see the inconsistency is to see which regularization is needed in order to show that $\delta(\epsilon_{\bar{z}})S = 0$. It turns out that one needs to define the following infinite summation:

$$\sum_{p=0}^{\infty} \left\{ \bar{\theta}_{\bar{z}}^{p,p} + \bar{\theta}_{\bar{z}}^{p+1,p+1} \right\} = \bar{\theta}_{\bar{z}}^{0,0}. \tag{A.23}$$

Like in the case of the toy model (see around eq. (A.20)) this regularization involves variables which are not gauge inert. We therefore encounter the same inconsistencies as in the toy model. The same applies to the superstring generalization of all other ($j > 2$) superparticle gauge symmetries, which we will not give explicitly here.

Thus, our conclusion is that the gauge fixed superstring action in a covariant gauge does not possess any *consistent* gauge symmetries.

References

- [1] R. Kallosh, Phys. Lett. B224 (1989) 273; B225 (1989) 49
- [2] M. Green and C. Hull, Phys. Lett. B225 (1989) 57
- [3] J. Gates, M. Grisaru, U. Lindström, P. van Nieuwenhuizen, M. Roček, W. Siegel and A. van de Ven, Phys. Lett. B225 (1989) 44;
U. Lindström, P. van Nieuwenhuizen, M. Roček, W. Siegel and A. van de Ven, Construction of the covariantly quantized heterotic superstring, University of Stony Brook preprint, ITP-SB-89-38 (1989)
- [4] U. Lindström, P. van Nieuwenhuizen, M. Roček, W. Siegel and A. van de Ven, Phys. Lett. B224 (1989) 285
- [5] W. Siegel, Talk presented at Strings '89, College Station, TX, March 13–18, 1989
- [6] I. Batalin and G. Vilkovisky, Phys. Rev. D28 (1983) 2567
- [7] C. Becchi, A. Rouet and R. Stora, Commun. Math. Phys. 42 (1975) 127;
I.V. Tyutin, P.N. Lebedev Physical Institute preprint N39 (1975)

- [8] P.K. Townsend, Phys. Lett. B90 (1980) 275;
W. Siegel, Phys. Lett. B93 (1980) 170;
P. van Nieuwenhuizen and E. Sezgin, Phys. Lett. B94 (1980) 179
- [9] J. Thierry-Mieg, BRS structure of the antisymmetric tensor gauge theories, Harvard preprint, HUMTP 79/B86, unpublished
- [10] R. Kallosh, Pis'ma Zh. Eksp. Teor. Fiz. 26 (1977) 575; Nucl. Phys. B141 (1978) 141
- [11] B. de Wit and J.W. van Holten, Phys. Lett. B69 (1979) 389
- [12] A. Batalin and G. Vilkovisky, Phys. Lett. B102 (1981) 27
- [13] D. Freedman, Cal. Tech. report CALT 68-624 (1977), unpublished;
P.K. Townsend, Local supersymmetry without gravity?, in Supergravity, ed. P. van Nieuwenhuizen and D. Freedman (North-Holland, Amsterdam, 1979);
D. Freedman and P. Townsend, Nucl. Phys. B177 (1981) 282
- [14] S. de Alwis, M. Grisaru and L. Mezincescu, Nucl. Phys. 303 (1988) 57;
L. Baulieu, E. Bergshoeff and E. Sezgin, Nucl. Phys. B307 (1988) 348;
C. Battle and J. Gomis, Phys. Rev. D38 (1988) 1169;
T. Clark, C. Lee and S. Love, Nucl. Phys. B308 (1988) 379
- [15] R. Kallosh, JETP Lett. 45 (1987) 365; Phys. Lett. B195 (1987) 369
- [16] E. Bergshoeff, E. Sezgin and P. Townsend, Phys. Lett. B189 (1987) 75;
A. Achúcarro, J. Evans, P. Townsend and D. Wiltshire, Phys. Lett. B198 (1987) 441
- [17] C. Thorn, Phys. Rep. 175 (1989) 1
- [18] M. Bocchichio, Phys. Lett. B188 (1987) 332; B193 (1987) 31
- [19] M.B. Green and J. Schwarz, Phys. Lett. B136 (1984) 367; Nucl. Phys. B243 (1984) 285
- [20] R. Kallosh and A. Morozov, Int. J. Mod. Phys. A3 (1988) 133; Phys. Lett. B207 (1988) 549
- [21] J. Thierry-Mieg, C. R. Acad. Sci. Paris 299 (1984) 1309
- [22] W. Siegel, Maryland preprint UMDEPP 88-231; Nucl. Phys. B284 (1987) 632
- [23] M. Green and C. Hull, The ghost structure of the superstring, Queen Mary College preprint QMC-89-11 (1989)
- [24] I. Bars and M. Günaydin, Commun. Math. Phys. 91 (1983) 31
- [25] J. Fisch and M. Henneaux, A note on the covariant BRST quantization of the super-particle, University of Bruxelles preprint, ULB TH2/89-04 (1989)
- [26] F. Bastianelli, W. Delius and E. Laenen, Free quantum action and BRST-charge for the superparticle, Stony Brook preprint, ITP-SB-89-51 (1989)